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JHEP08(2009)054

(http://iopscience.iop.org/1126-6708/2009/08/054)

Published by IOP Publishing for SISSA

Received: *June 24, 2009*
Accepted: *July 29, 2009*
Published: *August 14, 2009*

On walls of marginal stability in $\mathcal{N} = 2$ string theories

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Abstract: We study the properties of walls of marginal stability for BPS decays in a class of $\mathcal{N} = 2$ theories. These theories arise in $\mathcal{N} = 2$ string compactifications obtained as freely acting orbifolds of $\mathcal{N} = 4$ theories, such theories include the STU model and the FHSV model. The cross sections of these walls for a generic decay in the axion-dilaton plane reduce to lines or circles. From the continuity properties of walls of marginal stability we show that central charges of BPS states do not vanish in the interior of the moduli space. Given a charge vector of a BPS state corresponding to a large black hole in these theories, we show that all walls of marginal stability intersect at the same point in the lower half of the axion-dilaton plane. We isolate a class of decays whose walls of marginal stability always lie in a region bounded by walls formed by decays to small black holes. This enables us to isolate a region in moduli space for which no decays occur within this class. We then study entropy enigma decays for such models and show that for generic values of the moduli, that is when moduli are of order one compared to the charges, entropy enigma decays do not occur in these models.

Keywords: Black Holes in String Theory, Extended Supersymmetry, Supergravity Models

ArXiv ePrint: 0905.4115

[1]On lien from Harish-Chandra Research Institute, Allahabad, India.

doi:10.1088/1126-6708/2009/08/054

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1 Introduction

The spectrum of BPS states in a supersymmetric theory are known to jump across walls of marginal stability when asymptotic moduli are varied [1, 2]. The jumps in the spectrum across the line of marginal stability is given by the wall crossing formula [3, 4]. Any proposal for the BPS spectrum should incorporate these jumps. Recent studies have led to a good understanding of spectrum of $1/4$ BPS dyons in a large class of $\mathcal{N} = 4$ string theories [5–7].[1] All walls of marginal stability of $1/4$ BPS dyons with co-dimension one have been classified in these theories [9, 10] It has also been shown that jumps in the spectrum of $1/4$ BPS states across these walls are consistent with the wall crossing formula [9, 11–14]

For BPS states in $\mathcal{N} = 2$ theories a similar understanding has yet to emerge. In [15] a proposal for the spectrum of class of $1/2$ BPS state in the STU model was put forward. The first subleading corrections in entropy for large charges evaluated from this

[1] See [8] for a review.

proposal agrees with that evaluated using the Hawking-Bekenstein-Wald formula including the Gauss-Bonnet term. For large charges the partition function proposed for the STU model also reduces to the OSV form [16] on performing the Laplace transform with respect to the electric charges [17]. The pre-factor which arises in this Laplace transform agrees with that proposed by [18]. In [17], the proposed partition function for the STU model was argued to be to be valid only for single centered black holes. Thus the spectrum obtained from the proposed partition function is valid only when the asymptotic moduli equals the moduli at the attractor point. It is only for these values of the asymptotic moduli the single centered black hole is stable and multi-centered configurations do not exist. To understand how to extend the partition function to all regions in asymptotic moduli space it is necessary to study the walls of marginal stability in this model and the domains formed by the intersection of these walls.

To classify all the possible walls of marginal stability and study the domains formed by them for an arbitrary $\mathcal{N} = 2$ theory is in general a difficult task. This is because in $\mathcal{N} = 2$ theories all BPS decays are co-dimension one surfaces while in $\mathcal{N} = 4$ theories only decays to small black holes are co-dimension one surfaces [9, 10, 19]. In fact unless the surface of marginal stability is a co-dimension one surface, the index which counts the BPS state does not jump [20].

In this paper given a BPS state specified by a primitive charge vector (Q, P) corresponding to a large black hole we study various properties of walls of marginal stability in a class of $\mathcal{N} = 2$ theories. The properties we find will enable us to determine the conditions on the charges of the decay and the moduli such that we obtain simple domains in moduli space where states are stable under a class of decays. The class of $\mathcal{N} = 2$ models we will focus on in this paper are those theories constructed as freely acting orbifolds of $\mathcal{N} = 4$ theories. The vector multiplet moduli space of such theories is known exactly and is of the form

$$\mathcal{M}_V = \frac{SU(1,1)}{\mathrm{U}(1)} \times \frac{SO(2,n)}{SO(2) \times SO(n)}. \tag{1.1}$$

The axion-dilaton moduli τ, parametrizes the coset $SU(1,1)/\mathrm{U}(1)$, while the rest of the vector multiplets parametrizes the coset $SO(2,n)/(SO(2) \times SO(n))$. The STU model of [21, 22] has $n = 2$, while the FHSV [23] model has $n = 10$. Other models belonging to this class with $n = 4, 6$ have been constructed and studied in [24].

There is a convenient parametrization of the coset in (1.1) which enables us write down a simple mass formula for 1/2 BPS states in these models. From this mass formula we show that if the walls of marginal stability is seen as sections in the τ-plane they will be either circles or lines. Using the continuity properties of the walls of marginal stability we show that the central charges of larges black holes do not vanish in the interior of the moduli space. Since the walls are circles or lines in the τ plane it is sufficiently easy to study the intersection of these and find domains bounded by these walls. To isolate this class of decays we obtain certain properties of the walls of marginal stability which are true for all decays of a given charge vector in these theories. Here are some of the general properties of walls of marginal stability of a given charge vector (Q, P) we find in this paper.

1. All walls of marginal stability are circles or lines in the τ plane.

2. All walls of marginal stability meet at the same point in the lower half τ plane.

3. Walls of marginal stability corresponding to decays to two small black holes for any generic moduli always exist.

4. To specify a wall uniquely, it is just necessary to provide the two points r_- and r_+ at which it intersects the real axis in the τ plane.

5. The necessary and sufficient conditions that walls intersect each other only on the real axis and never in the interior of the upper half τ plane is

$$r_+ = \frac{p}{q}, \quad r_- = \frac{p'}{q'}, \quad \text{with} \quad pq' - p'q = 1,$$

   where $p, q, p', q' \in \mathbb{Z}$. If one of these points is at infinity, then the wall is line and it must pass through an integer on the real axis. In fact we show if the above conditions are not true, there is always a wall corresponding to a small black hole decay which intersects the circle joining r_- and r_+ or the lines passing through the integer points.

Let us now look at the structure of domains formed by walls of marginal stability. The structure of domains plays a role in determining or testing the BPS spectrum. If there is domain with no walls of marginal stability, then the BPS spectrum remains the same in that domain and does not jump. The boundaries of the domain will determine how the BPS spectrum jumps when one crosses the walls which determine the domain. An example of a simple domain bounded by walls of marginal stability is that found in [9] for the case of $\mathcal{N} = 4$ decays. This domain also exists for the $\mathcal{N} = 2$ models under consideration. If one restricts the attention to decays to two small black holes, the walls of marginal stability are circles or lines in the upper half τ plane. Examples of these are drawn in figure 1. The walls which correspond to decay to two small black holes are shown with bold lines. They intersect each other only on the real line and at at rational points. Consider the region A bounded by the line joining B and point -1 on the real line, the circle joining -1 and the origin and the line passing through the origin and D. In this region there are no decays to two small black holes. Similar domains exist in each interval $[n, n+1], n \in \mathbb{Z}$, for example the domain above the circle joining the origin and the point 1 on the real axis and so on. In this paper we isolate a class of decays, which include decays to large black holes whose walls of marginal stability always lies in the region II, that is in the region bounded by the small black hole decay corresponding to circles joining the points $(n, 0)$ and $(n+1, 0)$ on the real line.[2] These lines of marginal stability are represented by the dashed lines in figure 1. They can intersect each other or that of walls corresponding to small black decays in the interior of the upper half plane at the most once in region II. Thus restricting our attention to these class of decays, which include decays to large black holes we see that the region region A continues to be a domain where there are no decays occur among this class.

As we have a convenient mass formula for the 1/2 BPS states in these class of $\mathcal{N} = 2$ models, we can use it to study the entropy enigma found in [4, 25]. The enigma results

[2]The physical τ plane is determined by condition $\tau_2 \geq 0$.

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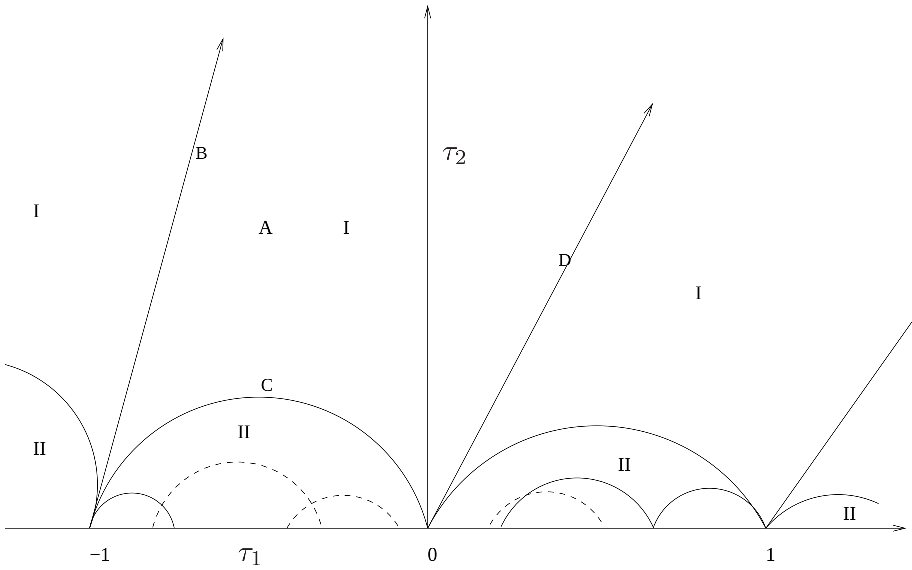

Figure 1. Walls of marginal stability in the upper half τ plane. Bold lines correspond to walls coincident with small black hole decays. Dashed line correspond to generic decays.

when a given 1/2 BPS states decays to products whose entropy is parametrically larger than that of the initial state. By analyzing the various cases we show that such decays are not allowed at generic regions of moduli space. By generic we mean when moduli are not scaled parmetrically and is of order one. This conclusion is consistent with that found in [4, 26] for specific examples, but here we demonstrate it in general for these class of $\mathcal{N} = 2$ models.

The organization of the paper is as follows: In section 2 we present the mass formula for 1/2 BPS states in the class of $\mathcal{N} = 2$ theories we will be dealing with and show that for BPS states, the mass or the central charge does not vanish in the interior of moduli space. We then present the conditions for the existence of walls of marginal stability and determine their equations. In section 3. we first show that all decays of a given charge vector meet at a common point in the lower half τ plane ($\tau_2 < 0$) . We then study small black hole decays of a given charge vector and show that walls corresponding to these decays always exist. We enumerate the conditions necessary for the existence of a wall corresponding to a generic decay. We find the necessary and sufficient conditions such two walls never meet in the interior of the upper half τ plane. We then determine the conditions which isolate a class of decays whose walls lie in region below that bounded by small black hole decays. In section 4 we study entropy enigma decays using the simple mass formula for these models and show that at generic regions of moduli these models do not admit such decays. The appendix contains some useful results from number theory necessary for our purposes.

2 BPS mass formula and marginal stability

In this paper we will be dealing with a class of $\mathcal{N}=2$ theories which are obtained by freely acting orbifolds of $\mathcal{N}=4$ theories. For these class of $\mathcal{N}=2$ theories, it is instructive to derive the mass formula for BPS states from the $\mathcal{N}=4$ mass formula. Since these theories are constructed from parent $\mathcal{N}=4$ models by a freely acting orbifold, the 1/4 BPS states as well as the 1/2 BPS states of the parent theory will be BPS states in the orbifolded $\mathcal{N}=2$ model. Therefore we can obtain the BPS mass formula in these models using the BPS mass formula of the $\mathcal{N}=4$ theory. Given a charge vector (Q,P) the BPS mass formula for $\mathcal{N}=4$ theories are given by [27, 28]

$$\begin{aligned} m(Q,P)^2 &= \frac{1}{\tau_2}(Q-\bar{\tau}P)^T(M+L)(Q-\tau P) \\ &\quad +2\left[(Q^T(M+L)Q)(P^T(M+L)P)-(P^T(M+L)Q)^2\right]^{1/2}. \end{aligned} \tag{2.1}$$

Note that in this case Q,P is a charge vector belonging to a Narain lattice with signature $\{(-1)^r,(1)^6\}$.[3] In (2.1) we have we have to choose the branch such that the square root is positive. This guarantees that the mass formula is given by the larger of the two central charges of the $\mathcal{N}=4$ model. M refers to the asymptotic moduli of the vector multiplets of the $\mathcal{N}=4$ theory and τ refers to the asymptotic dilaton-axion moduli. For the class of $\mathcal{N}=2$ models under consideration in this paper the vector multiplet moduli space is of the form

$$\begin{aligned} \mathcal{M}_V &= \mathcal{M}_S\times\mathcal{M}_T, \\ &= \frac{SU(1,1)}{\mathrm{U}(1)}\times\frac{SO(2,n)}{SO(2)\times SO(n)}. \end{aligned} \tag{2.2}$$

Thus to apply the mass formula in (2.1) to these $\mathcal{N}=2$ models we just have to restrict the matrix M so that it parametrizes the coset $SO(2,n)/(SO(2)\times SO(n)$ Therefore M is a $(2+n)\times(2+n)$ matrix which satisfies the conditions

$$M^T=M,\quad M^TLM=L, \tag{2.3}$$

where L is the diagonal$(2+n)\times(2+n)$ matrix given by

$$L=\mathrm{Dia}(-1^n,1^2).$$

The charge vectors for the $\mathcal{N}=2$ models take values in a Narain lattice with the same signature as L. We now proceed to give an explicit parametrization of M as in [15]. First introduce $n+2$ complex numbers w_I satisfying the constraint

$$-\sum_{I=1}^{n}w_I^2+w_{n+1}^2+w_{n+2}^2=0, \tag{2.4}$$

together with the identification $w_I\sim cw_I$, where c is a complex number. Note that the constraint in (2.4) and the identifications of w's up to complex scalings reduce the number

[3]For Heterotic on T^6, $r=22$

of independent parameters to $2n$ which is the required number of variables to parametrize the moduli space $\mathcal{M}_T$. Using the scaling degree of freedom, the constraints in (2.4) can be solved by introducing n complex numbers $(y^+, y^-, \vec{y})$ where $\vec{y}$ is a n dimensional vector. These variables are related to the w_I's by

$$\begin{aligned} w_I &= y_I, \qquad\qquad I = 1, \cdots n-2, \qquad w_{n-1} = \frac{1}{\sqrt{2}}(y^+ - y^-), \\ w_n &= 1 + \frac{y^2}{4}, \qquad w_{n+1} = \frac{1}{\sqrt{2}}(y^+ + y^-), \\ w_{n+2} &= -1 + \frac{y^2}{4}, \qquad y^2 = 2y^+y^- + \vec{y}^2. \end{aligned} \tag{2.5}$$

It is easy to see that these values of w_I satisfy the constraint in (2.4). The above parametrization amounts to scaling $w_n - w_{n+2}$ such that its value is fixed to be 2. Using the above solution of the constraint (2.4) it can be seen that

$$\begin{aligned} &-\sum_{I=1}^{n} |w_I|^2 + |w_{n+1}|^2 + |w_{n+2}|^2 = 2Y, \\ &\text{where} \quad Y = (\mathrm{Im} y)^2 = 2y_2^+ y_2^- - \vec{y}_2^2. \end{aligned} \tag{2.6}$$

Here the subscripts 2 in y refer to its imaginary part. Y is related to the Kähler potential on the moduli space by

$$K = -\log Y. \tag{2.7}$$

We can now parametrize the matrix M in (2.3) as follows

$$M = L\tilde{M}L - L, \qquad \tilde{M}_{IJ} = \frac{w_I \bar{w}_J + \bar{w}_I w_J}{Y}. \tag{2.8}$$

Using the above parametrization of M one can easily demonstrate its properties (2.3) using (2.4) and (2.6).

We use the above parametrization of the asymptotic moduli matrix M in terms of w's in the mass formula (2.1). It is first instructive to see what the terms in the square root in (2.1) reduces to

$$\begin{aligned} &2\left[(Q^T(M+L)Q)(P^T(M+L)P) - (P^T(M+L)Q)^2\right]^{1/2} \\ &\quad = 2\left[\frac{4|Q\cdot w|^2|P\cdot w|^2}{Y^2} - \frac{\{(Q\cdot w)(P\cdot\bar{w}) + (Q\cdot\bar{w})(P\cdot w)\}^2}{Y^2}\right]^{1/2}, \\ &\quad = \frac{2}{Y}\left[(-)\left\{(Q\cdot w)(P\cdot\bar{w}) - (Q\cdot\bar{w})(P\cdot w)\right\}^2\right]^{1/2}, \\ &\quad = \pm\frac{2i}{Y}\left[(Q\cdot w)(P\cdot\bar{w}) - (Q\cdot\bar{w})(P\cdot w)\right], \\ &\quad = \mp\frac{4}{Y}\mathrm{Im}((Q\cdot w)(P\cdot\bar{w})). \end{aligned} \tag{2.9}$$

Since we have to choose the positive square root we see that for $\mathrm{Im}((Q\cdot w)(P\cdot\bar{w})) < 0$ we have to choose the $-$ ve sign in the last line of (2.9) and for $\mathrm{Im}((Q\cdot w)(P\cdot\bar{w})) > 0$ we have

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to choose the + ve sign in the last line of (2.9). We summarize this in the equation below

$$\begin{aligned}&\left[(Q^T(M+L)Q)(P^T(M+L)P)-(P^T(M+L)Q)^2\right]^{1/2} \\ &\qquad = \frac{4}{Y}\left|\mathrm{Im}((Q\cdot w)(P\cdot\bar{w}))\right|.\end{aligned} \tag{2.10}$$

Let us now proceed and derive the BPS mass formula The first term in the mass formula (2.1) can be written as

$$\begin{aligned}&\frac{1}{\tau_2}(Q-\bar{\tau}P)^T(M+L)(Q-\tau P) \\ &\qquad = \frac{1}{\tau_2 Y}\left[((Q-\bar{\tau}P)\cdot w)((Q-\tau P)\cdot\bar{w})+((Q-\bar{\tau}P)\cdot\bar{w})((Q-\tau P)\cdot w)\right].\end{aligned} \tag{2.11}$$

Now adding (2.9) and (2.11) we obtain

$$m(Q,P)^2 = \frac{2}{\tau_2 Y}|(Q-\tau P)\cdot w|^2, \qquad \text{for } \mathrm{Im}((Q\cdot w)(P\cdot\bar{w}))<0, \tag{2.12}$$

$$m(Q,P)^2 = \frac{2}{\tau_2 Y}|(Q-\bar{\tau} P)\cdot w|^2, \qquad \text{for } \mathrm{Im}((Q\cdot w)(P\cdot\bar{w}))>0. \tag{2.13}$$

The first equality arises on choice of the − sign in (2.9), while the second equality arises on the choice of + sign in (2.9). For convenience let us define

$$Z(Q,P)=(Q-\tau P)\cdot w. \tag{2.14}$$

With this definition the BPS mass formula for the first case in (2.12) can be written as

$$m^2(Q,P)=|\tilde{Z}(Q,P)|^2=\frac{2}{\tau_2 Y}|Z(Q,P)|^2. \tag{2.15}$$

The above mass formula agrees with that found in [29] for $\mathcal{N}=2$ models with the vector multiplet moduli of the form given in (2.2)[4]

In a given $\mathcal{N}=2$ theory the mass of BPS states are given once and for all by one formula which is proportional to the absolute value of the central charge. Therefore we must choose one branch out of the two possible branches given in (2.12), (2.13). In this paper we will work with the first branch in (2.12), this implies that the condition $\mathrm{Im}((Q\cdot w)(P\cdot\bar{w}))<0$ must be true through out the moduli space for a given charge vector (Q,P). In section 2.2 we will demonstrate that once we choose to describe BPS states with the branch (2.12), then the condition $\mathrm{Im}((Q\cdot w)(P\cdot\bar{w}))<0$ remains true for BPS states for all asymptotic moduli w.

2.1 Marginal stability

In this section we state the conditions for marginal stability of a given BPS state and show that the co-dimension one surfaces seen as sections for fixed w moduli in the τ-plane

[4] See equation 5.28 of [29].

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are either circles or lines. Consider the marginal decay of a BPS primitive charge vector corresponding to a large black hole with $Q^2 > 0, P^2 > 0$ and $Q^2P^2 - (Q \cdot P)^2 > 0$[5]given by

$$\begin{pmatrix} Q \\ P \end{pmatrix} = \begin{pmatrix} Q_1 \\ P_1 \end{pmatrix} + \begin{pmatrix} Q_2 \\ P_2 \end{pmatrix}. \tag{2.16}$$

If the above decay is marginally allowed then the sum of the mass of the products equals the mass of the initial state, therefore we have the equation

$$m(Q, P) = m(Q_1, P_1) + m(Q_2, P_2) \tag{2.17}$$

Examining the mass formula in (2.15), we see that this implies the complex numbers

$$Z(Q, P), \quad Z(Q_1, P_1), \quad Z(Q_2, P_2),$$

are all co-linear. This leads to the following two conditions

$$\mathrm{Im}(Z_1 \bar{Z}_2) = 0, \tag{2.18}$$
$$\mathrm{Re}(Z_1 \bar{Z}_2) > 0, \tag{2.19}$$

where

$$Z_1 = (Q_1 - \tau P_1) \cdot w, \qquad Z_2 = (Q_2 - \tau P_2) \cdot w, \qquad Z = (Q - \tau P) \cdot w. \tag{2.20}$$

These two conditions are equivalent to the conditions that phases of the central charges align. The first equality imposes the condition that the phases are equal up to a multiple of π, this implies that imposing the first equality ensures that the phases can be aligned or off by π. While the second inequality assures that the phases align. Note that the first condition (2.18) can also be written equivalently as

$$\mathrm{Im}(Z\bar{Z}_1) = 0, \qquad \text{or} \quad \mathrm{Im}(Z\bar{Z}_2) = 0. \tag{2.21}$$

Let us now proceed to obtain the explicit equations in the τ plane for a given fixed w moduli. Substituting the definitions of Z in the (2.18) we obtain the following equation

$$\begin{aligned} &\tau\bar{\tau}\mathrm{Im}[P_1 \cdot w P_2 \cdot \bar{w}] - \tau_1 \mathrm{Im}[P_1 \cdot w Q_2 \cdot \bar{w} + Q_1 \cdot w P_2 \cdot \bar{w}] \\ &-\tau_2 \mathrm{Re}[P_1 \cdot w Q_2 \cdot \bar{w} - Q_1 \cdot w P_2 \cdot \bar{w}] + \mathrm{Im}[Q_1 \cdot w Q_2 \cdot \bar{w}] = 0 \end{aligned} \tag{2.22}$$

In general the above equation (2.22), is an equation of a circle in the τ-plane by completing the squares. To show this let us define

$$\begin{aligned} A &= \mathrm{Im}[P_1 \cdot w P_2 \cdot \bar{w}], & B &= \mathrm{Im}[P_1 \cdot w Q_2 \cdot \bar{w} + Q_1 \cdot w P_2 \cdot \bar{w}], \\ C &= \mathrm{Re}[P_1 \cdot w Q_2 \cdot \bar{w} - Q_1 \cdot w P_2 \cdot \bar{w}], & D &= \mathrm{Im}[Q_1 \cdot w Q_2 \cdot \bar{w}]. \end{aligned} \tag{2.23}$$

[5]In this paper we will be dealing only with primitive charge vectors, that is (Q, P) which cannot be written as an integral multiple of another vector.

Note that in these coefficients are unchanged when one replace $Q_1 \to Q, P_1 \to P$ or $Q_2 \to Q, P_2 \to P$ due to the property that the curve $\mathrm{Im}(Z\bar{Z}_1)$ can be written as (2.21). Completing the squares in (2.22) one obtains

$$\left(\tau_1 - \frac{B}{2A}\right)^2 + \left(\tau_2 - \frac{C}{2A}\right)^2 = \frac{B^2 + C^2 - 4AD}{4A^2}. \tag{2.24}$$

It can be shown using simple algebraic manipulations that

$$B^2 + C^2 - 4AD = |P_1 \cdot wQ_2 \cdot w - P_2 \cdot wQ_1 \cdot w|^2. \tag{2.25}$$

Thus for $A \neq 0$ the equation (2.22) is that of a circle, if $A = 0$ it reduces to that of the straight line.[6] Let us now examine the second condition for marginal stability given in (2.19). Again substituting the definition of Z into this condition we obtain the equation

$$\begin{aligned} &\tau\bar{\tau}\mathrm{Re}[P_1 \cdot wP_2 \cdot \bar{w}] - \tau_1 \mathrm{Re}[P_1 \cdot wQ_2 \cdot \bar{w} + Q_1 \cdot wP_2 \cdot \bar{w}] \\ &+\tau_2 \mathrm{Im}[P_1 \cdot wQ_2 \cdot \bar{w} - Q_1 \cdot wP_2 \cdot \bar{w}] + \mathrm{Re}[Q_1 \cdot wQ_2 \cdot \bar{w}] > 0. \end{aligned} \tag{2.26}$$

The curve determining the above inequality is also an equation of a circle, this can be seen by completing the squares.[7] Let us define the following

$$\begin{aligned} A' &= \mathrm{Re}[P_1 \cdot wP_2 \cdot \bar{w}], & B' &= \mathrm{Re}[P_1 \cdot wQ_2 \cdot \bar{w} + Q_1 \cdot wP_2 \cdot \bar{w}], \\ C' &= \mathrm{Im}[P_1 \cdot wQ_2 \cdot \bar{w} - Q_1 \cdot wP_2 \cdot \bar{w}], & D' &= \mathrm{Re}[Q_1 \cdot wQ_2 \cdot \bar{w}]. \end{aligned} \tag{2.27}$$

In terms of these variables, the inequality reduces to

$$A' \left[\left(\tau_1 - \frac{B'}{2A'}^2\right) + \left(\tau_2 + \frac{C'}{2A'}\right)^2 \right] - \frac{|P_1 \cdot wQ_2 \cdot w - P_2 \cdot wQ_1 \cdot w|^2}{4A'} > 0. \tag{2.28}$$

Here we have used the equality

$$(B')^2 + (C')^2 - 4A'D' = |P_1 \cdot wQ_2 \cdot w - P_2 \cdot wQ_1 \cdot w|^2. \tag{2.29}$$

Therefore the wall of marginal stability is given by the part of the circle in (2.22) which satisfies the inequality in (2.26) and which lies in the physical part of the τ plane. We call such a wall of marginal stability as a the wall corresponding to a physical decay. Let us determine the point of intersection of the circle which determines the inequality in (2.26) and the circle in (2.22). From (2.18) and (2.19), we see that they intersect at the two points

$$Z_1 = 0, \quad \text{or} \quad Z_2 = 0. \tag{2.30}$$

We will show in section 2.2 that these points of intersection never lie in the interior of the physical moduli space, that is in the interior of the upper half τ plane ($\tau_2 > 0$).

[6]It can be shown that if one examines the equation $\mathrm{Im}(Z_1\bar{Z}_2)$ for constant $\tau, y^-, \vec{y}$ moduli in the y^+ plane, the resultant curve is also a circle. Similarly the curve is also a circle in the y^- plane for constant $\tau, y^+, \vec{y}$.

[7]The inequality $\mathrm{Re}(Z_1\bar{Z}_2) > 0$ in the y_+ plane for constant $\tau, y^-, \vec{y}$ moduli is also determined by a circle. The same statement can be made when the inequality is seen in the y^- plane for constant $\tau, y^+, \vec{y}$.

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Let us now compare the conditions for walls of marginal stability with that for the existence of the two centered black hole solution with charges (Q_1, P_1), (Q_2, P_2). The integrability condition [3] for the existence of the two centred solution results in the following equation

$$R = \frac{(Q_1 \cdot P_2 - Q_2 \cdot P_1)|\tilde{Z}(Q,P)|}{\mathrm{Im}(Z_1 \bar{Z}_2)}. \tag{2.31}$$

where R is the distance between the two centers. Therefore we see that generically, when $(Q_1 \cdot P_2 - Q_2 \cdot P_1) \neq 0$ the two centred solutions are stable and exist at one side of the line $\mathrm{Im}(Z_1 \bar{Z}_2) = 0$ and are unstable on the other side. At the wall $\mathrm{Im}(Z_1 \bar{Z}_2) = 0$ the distance between the two centers of the black holes goes to infinity.

2.2 Non-vanishing of central charges and $\mathrm{Im} Q \cdot wP \cdot \bar{w} < 0$

It has been argued in [1, 30, 31] that the central charge of a BPS state corresponding to a large black hole never vanishes in the interior of the physical moduli space. In fact, it has been shown that the minimum value of the modulus of the central charge of a given BPS state occurs at the attractor value of the moduli which is proportional to the classical entropy of the corresponding black hole [31]. The fact that the central charge of a BPS state never vanishes at the interior of the physical moduli space was used recently in [32] in studying decay of D0-D6 bound states. For small black holes the central charge does vanish, but as we will see later, this occurs at rational and real values of the axion dilaton moduli not in the interior of the moduli space. If the central charged does vanish in the interior of the moduli space, then the corresponding state is not BPS.

In this section we provide an alternate argument that the central charges do not vanish in the interior of the moduli space based on the continuity properties of the walls of marginal stability. This will also enable use to derive the fact that the sign of $\mathrm{Im}(Q \cdot wP \cdot \bar{w})$ is maintained through out the moduli space. Let us suppose the central charge say $Z_1 = Z(Q_1, P_1)$ vanishes in the interior of the moduli space. Consider any decay in which Z_1 is one of the decay products, we then have

$$Z(Q,P) = Z(Q_1, P_1) + Z(Q_2, P_2). \tag{2.32}$$

The wall of marginal stability in the upper half τ plane is determined by the circle (2.18) and the inequality (2.19). As discussed earlier, the two circles intersect at (2.30), that is either at $Z_1 = 0$ or $Z_2 = 0$. Now that since the point $Z_1 = 0$ lies in the interior of the moduli space, we will have a situation which is schematically shown in figure 2. In this figure the circle with the bold line corresponds to the equation $\mathrm{Im}(Z_1 \bar{Z}_2) = 0$, while the circle with the the dashed lines correspond to the equation $\mathrm{Re}(Z_1 \bar{Z}_2) = 0$. They intersect at point A. For definiteness let the exterior of the circle with dashed lines be the domain in which $\mathrm{Re}(Z_1 \bar{Z}_2) > 0$. Then the arc ABC is the segment along which both the equation (2.18) and the inequality (2.19) are satisfied. Along the arc DA the equation (2.18) is satisfied while the inequality is not. Therefore along the arc ABC the charges Z_1 and Z_2 are aligned, while along the arc DA, the charges are anti-aligned. Now the equation for the stability of the two centered black hole is given by (2.31). From this we see that two centered black hole solution will be stable at one side of the circle with bold lines. Let us assume that for definiteness it is stable in the interior of the circle with bold lines, that is (2.18). This results in the following contradition: Consider a two centered black hole solution along the arc DA, just in the interior on the bold circle. The distance between the centres of the black holes from (2.31) is infinite. Therefore they do not interact with each other. Since they don't interact the total mass of the system is just sum of the masses of the two black holes.[8] Therefore the mass of the corresponding single centred black hole from which the two centred black hole has decayed is given by

$$|Z(Q,P)| = |Z(Q_1,P_1)| + |Z(Q_2,P_2)|. \tag{2.33}$$

But from the fact along the arc DA we have the condition $\mathrm{Re}(Z_1\bar{Z}_2) < 0$, the central charges are anti-aligned along DA. Thefore we have the equation

$$|Z(Q,P)| = ||Z(Q_1,P_1)| - |Z(Q_2,P_2)||. \tag{2.34}$$

From equations (2.33) and (2.34) we have obtained a contradition. Thus we see that the central charges cannot vanish in the interior of moduli space. A similar contradiction can be obtained if the two centered black hole is stable in the exterior of the circle with bold lines and also if $\mathrm{Re}(Z_1\bar{Z}_2) > 0$ is satisfied in the interior of the circle with dashed lines.

As mentioned earlier, there are cases in which the central charge can vanish at the boundaries of moduli space. These occur for small black holes. Consider the case in which the electric and the magnetic charges are proportional $(Q,P) = (mM, nM)$, Then the central charge vanishes at $\tau = \frac{m}{n}$ with $M \cdot w \neq 0$. This point is at the boundary of moduli space, one can map it to infinity by a $SL(2,\mathbb{Z})$ transformation. It is easy to see that the above argument does not apply to such states, since the point A is on the real line for these states not in the interior of the moduli space. In fact these states are dual to purely electric states with electric charge M. The condition $M \cdot w \neq 0$ restricts the argument from being applied to gauge bosons which can become massless at special points in moduli space. Gauge bosons satisfy the condtion $Q^2 = -1, P = 0$, which are excluded from our analysis.

Sign of $\mathrm{Im}(Q \cdot w P \cdot \bar{w})$. Now that we know the central charge of any BPS state does not vanish in the interior of the moduli space we can show that the the condition

$$\mathrm{Im}[(Q \cdot w)(P \cdot \bar{w})] < 0$$

holds through in the interior of the moduli space. Let us suppose that there are regions in the moduli space where the above condition is violated. Let $\tilde{w}$ be in such a region, then

$$\mathrm{Im}(Q \cdot \tilde{w})(P \cdot \bar{\tilde{w}}) > 0 \tag{2.35}$$

Let us examine the central charge at the point

$$\tilde{\tau}_2 = \frac{\mathrm{Im}(Q \cdot \tilde{w})(P \cdot \bar{\tilde{w}})}{|P \cdot \tilde{w}|^2}, \qquad \tilde{\tau}_1 = \frac{\mathrm{Re}(Q \cdot \tilde{w})(P \cdot \bar{\tilde{w}})}{|P \cdot \tilde{w}|^2}. \tag{2.36}$$

[8] The author thanks Ashoke Sen for pointing this important fact.

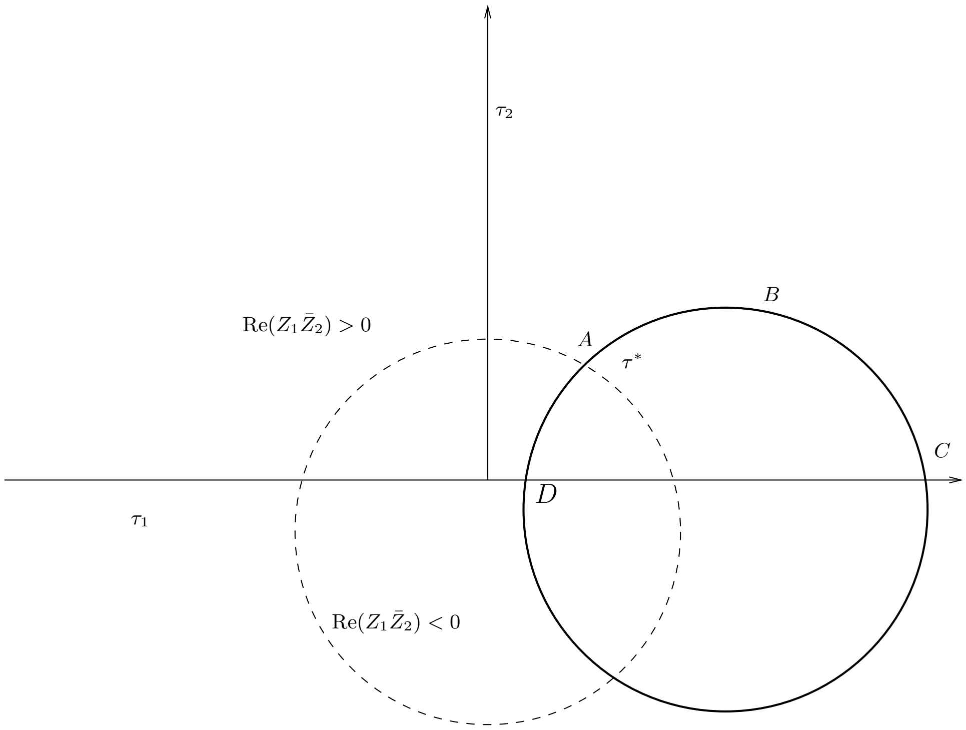

Figure 2. Contradition obtained if the central charge Z_1 vanishes in the interior of the moduli space: The inequalities indicate the central charges Z_1 and Z_2 are anti-aligned along DA. But since the two black holes are at infinite distance along DA the masses just add, therefore the central charges must be aligned.

This point certainly lies in the interior of the moduli space due to (2.35). At this point the central charge

$$Z(Q, P; \tilde{\tau}) = (Q \cdot \tilde{w} - \tilde{\tau} P \cdot \tilde{w}) = 0. \tag{2.37}$$

We have arrived at a contradition. Therefore there are no regions in the interior of moduli space for which the condition $\mathrm{Im}[(Q \cdot w)(P \cdot \bar{w})] < 0$ is violated. Thus once the branch (2.12) for the BPS mass formula is chosen, it does not change over the entire moduli space.

3 Properties of walls of marginal stability

To get an idea of the geometric structure of the domains bounded by the various lines of marginal stability we need to study the possible intersection points of these walls. We first consider two possible decays of a given charge vector (Q, P) corresponding to a BPS state.[9] We denote the decays by the following equations

$$Z(Q, P) = Z(Q_1, P_1) + Z(Q_2, P_2), \qquad Z(Q, P) = Z(Q_1', P_1') + Z(Q_2', P_2'). \tag{3.1}$$

[9] (Q, P) belong to the Narain lattice of the respective $\mathcal{N} = 2$ model. We also choose to work in some duality frame in which $P \neq 0$

For ease of notation we define

$$Z_1 = Z(Q_1, P_1), \quad Z_2 = Z(Q_2, P_2), \quad Z_1' = Z(Q_1', P_1'), \quad Z_2' = Z(Q_2', P_2'). \tag{3.2}$$

The equations that determine the two walls of marginal stability are given by

$$\begin{aligned} \mathrm{Im}(Z_1\bar{Z}) &= 0, \qquad \mathrm{Re}(Z_1\bar{Z}) > 0, \\ \mathrm{Im}(Z_1'\bar{Z}) &= 0, \qquad \mathrm{Re}(Z_1'\bar{Z}) > 0. \end{aligned} \tag{3.3}$$

Here we have used the equivalent form given in (2.21) to write the equations for the walls of marginal stability. From our earlier discussion, the walls of marginal stability of each of the two decays are determined by the equality together with the restriction obtained from the corresponding inequality. We will examine these walls as sections in the τ plane for a given moduli w. Therefore the physically relevant part of the wall is that part which lies in the domain $\tau_2 \geq 0$.

We organize this section as follows: In section 3.1 we show that all walls of marginal decay of a given charge vector (Q, P) meet at a point in the lower half τ plane ($\tau_2 < 0$). We then discuss the structure of the walls due to the decay of a given charge vector to two small black holes. In section 3.3 we show that a wall can be characterized by two real numbers r_+ and r_- at which the circle intersects the real axis in the τ plane. In section 3.4 we study walls formed by a generic black hole decay and state the conditions on the moduli and the charges which ensure a given decay physical. In section 3.5 we use the fact that all walls meet at a point in the lower half of the τ plane to to determine the intersection point of a wall corresponding to a decay to small black holes and any generic decay. In section 3.6 we show that the only walls which don't intersect each other in the interior of the τ plane are the ones corresponding to the small black hole decay or the ones whose walls coincide with the walls of small black hole decays. In section 3.7 we isolate the conditions on the moduli and the charges of a generic decay so that the corresponding wall always lies in a region bounded by the walls corresponding to small black hole decays.

3.1 All walls meet at the same point in the lower half τ plane

We know that the equations $\mathrm{Im}(Z_1\bar{Z}) = 0$ and $\mathrm{Im}(Z_1'\bar{Z}) = 0$ are generically circles. Therefore they intersect each other at the most twice. We now observe that a point common to these circles is the point τ^* where Z vanishes. This point is given by

$$\tau_1^* = \frac{\mathrm{Re}(Q \cdot wP \cdot \bar{w})}{|P \cdot w|^2}, \qquad \tau_2^* = \frac{\mathrm{Im}(Q \cdot wP \cdot \bar{w})}{|P \cdot w|^2}. \tag{3.4}$$

Since $\mathrm{Im}(Q \cdot wP \cdot \bar{w}) < 0$ for all w, this point lies in the lower half τ plane. Thus we conclude that all walls corresponding to decays of a given charge vector (Q, P) meet at the point (3.4) in the lower half τ plane. Though this point is not physically relevant we will see that it provides us useful information about the geometric structure of the domains formed by the walls. In fact from the knowledge of this point it is easy to determine the possible second intersection point of the walls for a generic pair of decays. It is possible for the second point to be in a valid region of the moduli space and also satisfy the inequality

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in (3.3). As we will see this second point determines the structure of the domains formed by the various walls.

Let us now restrict our attention to the situation in which one of the decays is to two small black holes in (3.3) and the other decay is more generic, including decays to large black holes. We start with examining the wall corresponding to the decay to small black holes.

3.2 Walls for decay into two small black holes

By small black holes we mean those BPS states in these $\mathcal{N}=2$ theories whose electric and magnetic charges are proportional. The charge vector (Q,P) is given by (mM,nM) where M is a given vector in the Lorentzian lattice and m,n are integers.[10] We first study the lines of marginal stability for decay to two small black holes. We can then parametrize the decay as in [9]

$$\begin{pmatrix} Q \\ P \end{pmatrix} = \begin{pmatrix} adQ - abP \\ cdQ - cbP \end{pmatrix} + \begin{pmatrix} -bcQ + abP \\ -cdQ + adP \end{pmatrix}, \tag{3.5}$$

where $ad-bc=1$ and $\{a,b,c,d\}\in\mathbb{Z}$. For simplicity we have assumed that the S-duality symmetry of the theory is $SL(2,\mathbb{Z})$. In the freely acting orbifold construction of these theories given in [21–24] the S-duality group is usually a subgroup of $SL(2,\mathbb{Z})$. In that case quantization of the charges leads to the condition: $\{a,b,d\}\in\mathbb{Z}$ while $c\in N\mathbb{Z}$ for the S-duality group $\Gamma_1(N)$ [9]. One can easily generalize the conclusions found in this paper for these cases. As shown in [9], the above parametrization is a unique parametrization of the decay into two small black holes up to the following transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.6}$$

Substituting the parametrization given in (3.5) into the conditions for marginal stability (2.18) and (2.19), we obtain the following equation and inequality.

$$\begin{aligned} &\mathrm{Im}(Q\cdot wP\cdot\bar{w})\left(cd|\tau|^2-(bc+ad)\tau_1+ab\right) \\ &\quad -(cd|Q\cdot w|^2+ab|P\cdot w|^2-(ad+bc)\mathrm{Re}(Q\cdot wP\cdot\bar{w}))\tau_2=0, \\ &\left(-cd|Q\cdot w|^2-ab|P\cdot w|^2+(bc+ad)\mathrm{Re}(Q\cdot wP\cdot\bar{w})\right)\left(cd|\tau|^2-(ad+bc)\tau_1+ab\right) \\ &\quad -\mathrm{Im}(Q\cdot wP\cdot\bar{w})\tau_2>0. \end{aligned} \tag{3.7}$$

It is convenient to perform a duality transformation to convert the conditions of marginal stability to straight lines. We consider the following S-duality transformations

$$\tau = \frac{a\tau'+b}{c\tau'+d}, \qquad \begin{pmatrix} Q \\ P \end{pmatrix} = \begin{pmatrix} a\tilde{Q}+b\tilde{P} \\ c\tilde{Q}+d\tilde{P} \end{pmatrix} \tag{3.8}$$

[10] There can be other classes of small black holes, for instance in the STU model, one can look at states whose electric and magnetic charges are proportional in the T frame rather than the usual S frame. Our discussion will apply to these decays if one examines the wall in the y^+ plane for a given τ, y^- moduli.

Then in terms of these new variables the equations reduce to

$$\begin{aligned} -\operatorname{Im}(\tilde{Q}\cdot w\tilde{P}\cdot\bar{w})\tau_1' + \operatorname{Re}(\tilde{Q}\cdot w\tilde{P}\cdot\bar{w})\tau_2' &= 0 \\ -\operatorname{Re}(\tilde{Q}\cdot w\tilde{P}\cdot\bar{w})\tau_1' - \operatorname{Im}(\tilde{Q}\cdot w\tilde{P}\cdot\bar{w})\tau_2' &> 0. \end{aligned} \tag{3.9}$$

Now substituting for τ_1' from the first equation into the second inequality one obtains

$$-\frac{|\tilde{Q}\cdot w\tilde{P}\cdot w|^2}{\operatorname{Im}(\tilde{Q}\cdot w\tilde{P}\cdot\bar{w})}\tau_2' > 0. \tag{3.10}$$

Since we have to examine the wall only in the physical τ plane, we have $\tau_2' \geq 0$. This implies that the phases align on the line of marginal stability only if $\operatorname{Im}(\tilde{Q}\cdot w\tilde{P}\cdot\bar{w}) < 0$. It is easily seen that this condition is indeed true, since

$$\begin{aligned} \operatorname{Im}[\tilde{Q}\cdot w\tilde{P}\cdot\bar{w}] &= \operatorname{Im}[(dQ-bP)\cdot w(-cQ+aP)\cdot\bar{w}], \\ &= \operatorname{Im}(Q\cdot wP\cdot\bar{w}) < 0. \end{aligned} \tag{3.11}$$

Where we have substituted for $(\tilde{Q},\tilde{P})$ in terms of (Q,P) from (3.8) and used the condition $ad-bc=1$. As we have shown that the the condition $\operatorname{Im}(Q\cdot wP\cdot\bar{w}) < 0$ is maintained throughout the interior of the moduli space, the inequality (3.10) is always satisfied in the upper half τ plane. This implies that the part of the line given in (3.9) in the upper half plane is the wall of marginal stability for the decay of the charge vector (Q,P) to small black hole given by equation (3.5) *Therefore given the charge vector (Q,P) and the moduli w there always exists a wall of marginal stability for the charge vector to decay into small black holes.*

The analysis of the structure of the domains formed by these walls was done in [9] for $\mathcal{N}=4$ theories. The same analysis goes through for these class of $\mathcal{N}=2$ models. A schematic diagram of the domains is given in figure 1. The small black hole decays consists of the bold curves. There are lines passing through the integer points on the real axis and there are circles joining each of these points. Then there are other circles which always lie below the circles joining the integer points. None of the circles corresponding to small black hole decay intersect each other in the interior of the upper half plane. The circles can intersect in the physical part of the τ plane only on the real axis. From (3.7) we see that the wall of marginal stability intersects the real axis in the τ plane at

$$r_+ = \frac{a}{c}, \quad \text{and,} \quad r_- = \frac{b}{d}, \quad \text{with} \quad ad-bc=1, \tag{3.12}$$

and it is only these points that can be possible meeting points of the walls corresponding to two different small black hole decays.

3.3 Characterization of a generic decay

We have seen in the previous section that small black holes decays are characterized by the integers a,b,c,d with $ad-bc=1$. In this section we introduce a simple method of characterization of the wall for a general decay.. This characterization enables one to easily

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determine if two walls intersect each other in the interior of the upper half plane. As we will see, this enables one to easily classify walls. Consider a generic decay given by

$$\begin{pmatrix} Q \\ P \end{pmatrix} = \begin{pmatrix} Q'_1 \\ P'_1 \end{pmatrix} + \begin{pmatrix} Q'_2 \\ P'_2 \end{pmatrix} \tag{3.13}$$

From (2.18) and (2.19) it is easy to see that the wall of marginal stability of this decay is determined by the following

$$\begin{aligned} &\tau\bar{\tau}\mathrm{Im}[P'_1 \cdot wP \cdot \bar{w}] - \tau_1 \mathrm{Im}[(P'_1 \cdot wQ \cdot \bar{w}) + (Q'_1 \cdot wP \cdot \bar{w})] \\ &\quad -\tau_2 \mathrm{Re}[(P'_1 \cdot wQ \cdot \bar{w}) - (Q'_1 \cdot wP \cdot \bar{w})] + \mathrm{Im}[Q'_1 \cdot wQ \cdot \bar{w}] = 0 \end{aligned} \tag{3.14}$$

$$\begin{aligned} &\tau\bar{\tau}\mathrm{Re}[P'_1 \cdot wP'_2 \cdot \bar{w}] - \tau_1 \mathrm{Re}[(P'_1 \cdot wQ'_2 \cdot \bar{w}) + (Q'_1 \cdot wP'_2 \cdot \bar{w})] \\ &\quad +\tau_2 \mathrm{Im}[(P'_1 \cdot wQ'_2 \cdot \bar{w}) - (Q'_1 \cdot wP'_2 \cdot \bar{w})] + \mathrm{Re}[Q'_1 \cdot wQ'_2 \cdot \bar{w}] > 0 \end{aligned} \tag{3.15}$$

We now wish to state the conditions under which the above wall of marginal stability is physical. That is, the conditions that ensure that the part of the circle in (3.14) lies in the upper half τ-plane and this part of the circle satisfies the inequality in (3.15). For ease of notation let us again define the coefficients

$$\begin{aligned} A &= \mathrm{Im}[P'_1 \cdot wP \cdot \bar{w}], & A' &= \mathrm{Re}[P'_1 \cdot wP'_2 \cdot \bar{w}], \\ B &= \mathrm{Im}[P'_1 \cdot wQ \cdot \bar{w} + Q'_1 \cdot wP \cdot \bar{w}], & B' &= \mathrm{Re}[P'_1 \cdot wQ'_2 \cdot \bar{w} + Q'_1 \cdot wP'_2 \cdot \bar{w}], \\ C &= \mathrm{Re}[P'_1 \cdot wQ \cdot \bar{w} - Q'_1 \cdot wP \cdot \bar{w}], & C' &= -\mathrm{Im}[P'_1 \cdot wQ'_2 \cdot \bar{w} - Q'_1 \cdot wP'_2 \cdot \bar{w}], \\ D &= \mathrm{Im}[Q'_1 \cdot wQ \cdot \bar{w}], & D' &= \mathrm{Re}[Q'_1 \cdot wQ'_2 \cdot \bar{w}]. \end{aligned} \tag{3.16}$$

We already know that the circle (3.14) passes through the point τ^* given in (3.4) which lies in the lower half τ plane. Therefore, the necessary condition one needs to impose so that part of the circle in (3.14) lies in the upper half plane is that it should intersect the real axis. This is given by the following

$$\begin{aligned} &B^2 - 4AD > 0, \quad \text{or} \\ &(\mathrm{Im}[P'_1 \cdot wQ \cdot \bar{w} + Q'_1 \cdot wP \cdot \bar{w}])^2 - 4\mathrm{Im}[P'_1 \cdot wP \cdot \bar{w}]\mathrm{Im}[Q'_1 \cdot wQ \cdot \bar{w}] > 0. \end{aligned} \tag{3.17}$$

We also must make sure that if (3.14) is to be a physical line of marginal stability, the points on the curve in (3.14) must also satisfy the inequality (3.15). We will discuss the various conditions which ensure this in the next section. We now introduce a method to characterize the circles in (3.14) provided it satisfies the condition (3.17). Instead of choosing A, B, C, D to specify this circle we characterize the circle as follows: We already know that this circle passes through the common point (3.4). This point is completely specified by the initial charge vector (Q, P) and the moduli w. We need two more points to characterize this circle. Due to the condition (3.17), it is clear that it intersects the real axis. The points of intersection are given by

$$r_\pm = \frac{B \pm \sqrt{B^2 - 4AD}}{A}. \tag{3.18}$$

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Without loss of generality we will assume $r_- < r_+$, the case when $r_- = r_+$ just results in the circle being tangent to the real line from below. Since the point (3.4) is common to all possible decays, specifying the two points of intersection on the real axis r_+ and r_-4 completely determines the circle. Note that when $A \to 0$, $r_+ \to \infty$, then the circle reduces to a line and r_- then refers to the point of intersection of the line with the real axis. Though one has uniquely specified the circle using these numbers, the decay which corresponds to a circle specified by r_- and r_+ is not unique. A decay with the parameters $\lambda A, \lambda B, \lambda C, \lambda D$, will also have the same values of r_+ and r_- and pass through (3.4). The advantage of this characterization of the circle is that given a pair of circles (r_-, r_+) and (r'_-, r'_+) they intersect in the interior of the upper half plane if and only if either of the following conditions are satisfied

$$r_- < r'_- < r_+ < r'_+, \quad \text{or} \quad r'_- < r_- < r'_+ < r_+. \tag{3.19}$$

3.4 Existence conditions for walls of generic decays

In this section we will find the conditions necessary so that the part of the circle (3.14) which emerges in the upper half plane also satisfies the inequality (3.15). Let us examine the situation when $A = \mathrm{Im}(P'_1 \cdot wP \cdot \bar{w}) \neq 0$. Then we can use the equation in (3.14) to write the inequality in (3.15) as

$$\frac{A'B - AB'}{A}\tau_1 + \frac{(A'C - AC')}{A}\tau_2 + \frac{(AD' - A'D)}{A} > 0. \tag{3.20}$$

We also know from the discussion above (2.30) that the line which determines the inequality in (3.20) intersects the circle (3.14) only in the lower half τ plane. This is because the intersection of the line in (3.20) and the circle (3.14) occurs at points where the central charges Z'_1 or Z'_2 vanishes and these are only in the lower half τ plane. Therefore, there are no other points at which the line (3.20) intersects the circle (3.14). This implies that to ensure that the part of the circle which emerges in the upper half plane satisfies the inequality it is sufficient to demand that any point on the circle (3.14) satisfies the inequality (3.20). We know that $\tau = (r_+, 0)$ and $\tau = (r_-, 0)$ are points on the circle (3.14). Thus the average of these points also must satisfy the inequality (3.20). This gives the condition

$$\begin{aligned}(r_+ + r_-)\frac{A'B - AB'}{A} + 2\frac{AD' - A'D}{A} > 0,\\ B(A'B - AB') + 2(AD' - A'D)A > 0,\end{aligned} \tag{3.21}$$

where we have used $r_+ + r_- = B/A$. Note that this true only if $A'B - AB' \neq 0$. Rewriting the above condition in terms of the values of the coefficients and after some simplifications we obtain

$$\begin{aligned}&\left[|P'_1 \cdot w|^2 \mathrm{Im}(P'_2 \cdot wQ'_2 \cdot \bar{w}) + |P_2 \cdot w|^2 \mathrm{Im}(P'_1 \cdot wQ'_2 \cdot \bar{w})\right]\\ &\quad \times \mathrm{Re}(P'_1 \cdot wQ_2 \cdot \bar{w} + Q'_1 \cdot wP'_2 \cdot \bar{w}) - 2(P'_1 \cdot wP'_2 \cdot \bar{w})\mathrm{Im}(P'_1\bar{w}P_2 \cdot wQ'_1 \cdot wQ_2 \cdot \bar{w}) > 0.\end{aligned} \tag{3.22}$$

For completeness let us discuss the case in which $A'B - AB' = 0$, evaluating this explicitly we obtain

$$A'B - AB' = |P'_1 \cdot w|^2 \mathrm{Im}(P'_2 \cdot wQ'_2 \cdot \bar{w}) + |P_2 \cdot w|^2 \mathrm{Im}(P'_1 \cdot wQ'_2 \cdot \bar{w}), \tag{3.23}$$

Since both the terms are of the same sign, this can vanish only if $\mathrm{Im}(P_2' w Q_2 \bar{w}) = \mathrm{Im}(P_1' \bar{w} Q_1' \bar{w}) = 0$. This occurs when both the decay products are small black holes. Now

$$AD' - A'D = -\mathrm{Im}(P_1' \bar{w} P_2 \cdot w Q_1' \cdot w Q_2 \cdot \bar{w})$$

which also vanishes for small black hole decays. Thus the condition (3.21) for small black hole decay reduces to

$$\frac{A'C - AC'}{A} > 0 \quad \text{if} \quad A \neq 0.. \tag{3.24}$$

We can verify this condition for the existence of a physical wall of marginal for small black hole decay. Evaluating the condition (3.17) for this case by substituting the values of the charges of the decay products from (3.7) and evaluating the coefficients from either (3.16) or (3.7) we obtain

$$B^2 - 4AD = [\mathrm{Im}(Q \cdot wP \cdot \bar{w})]^2 (bc - ad)^2 > 0, \tag{3.25}$$

which is always satisfied. Furthermore we see that $\mathrm{Im}(P_1' \cdot wP \cdot \bar{w}) = cd\mathrm{Im}(Q \cdot wP \cdot \bar{w}) \neq 0$ if $cd \neq 0$. Now examining the condition in (3.22) we see

$$-\frac{\left\{-cd|Q \cdot w|^2 - ab|P \cdot w|^2 + (bc + ad)\mathrm{Re}(Q \cdot wP \cdot \bar{w})\right\}^2 + [\mathrm{Im}(Q \cdot wP \cdot \bar{w})]^2}{\mathrm{Im}(Q \cdot wP \cdot \bar{w})} > 0. \tag{3.26}$$

It is now clear that the wall is physical when $\mathrm{Im}(Q \cdot wP \cdot \bar{w}) < 0$ which as we have seen is always true. Thus the decay to two small black holes is always allowed, which is the same conclusion reached earlier.

Let us now consider the case of $A = \mathrm{Im}(P_1' \cdot wP \cdot \bar{w}) = 0$. Note that this condition in general imposes an additional condition on the w moduli. But then the wall of marginal stability we will obtain is a higher co-dimension surface. Therefore we must look for situations when A vanishes at generic w moduli. This occurs if $P_1' = 0$ or if $P_1' = \alpha P$ and $\alpha \neq 1$.[11] Let us first examine the situation when $P_1' = \alpha P$. Now in this case, the circle in (3.14) reduces to a line.

$$-B\tau_1 - C\tau_2 + D = 0, \tag{3.27}$$

while the coefficient A' reduces to $\alpha(1-\alpha)|P \cdot w|^2$. We know the intersection of the circle which determines the inequality in (3.15) and the line in (3.27) lies in the lower half plane. If the points on the line in (3.27) satisfies the inequality (3.15), then it must be true that the inequality must hold when $\tau_2 \to \infty$. Thus if the coefficient $A' > 0$ the inequality (3.15) is satisfied and the resulting decay is physical. But since $A' = \alpha(1-\alpha)|P \cdot w|^2$ we must have $0 < \alpha < 1$ for $A' > 0$. Now let us examine the situation when $P_1 = 0$. For this case we see that the coefficient A' also vanishes. Thus both the circle in (3.14) and the circle determining the inequality (3.15) reduces to following lines.

$$-B\tau_1 - C\tau_2 + D = 0, \qquad -B'\tau_1 - C'\tau_2 + D' > 0. \tag{3.28}$$

[11] $P \neq 0$ by assumption, $P_1' = P$ is the same physical situation as $P_1' = 0$, because in this case $P_2' = 0$

Case (i)	$A \neq 0, A'B - AB' \neq 0$ and	
	$(A'B - AB')B + 2(AD' - A'D)A > 0$	See (3.22)
Case (ii)	$A = 0, P_1' = 0, B, \neq 0$ and $\frac{B'C - BC'}{B} > 0$	See (3.31)
	$A = 0, P_1' = \alpha P$ and $0 < \alpha < 1$, $\Rightarrow A' > 0$,	Decay allowed
Case (iii)	Small black hole decay $A'B - AB' = 0$	Always allowed

Table 1. Conditions for the existence for a wall of marginal stability

Let us suppose $B = \mathrm{Im}(Q_1' \cdot wP \cdot \bar{w}) \neq 0$. The lines intersect at a point in the lower half plane. This is the point at which one of the central charges vanish. Eliminating τ_1 from the first equation one gets the condition

$$\frac{B'C - BC'}{B}\tau_2 - \frac{B'D + BD'}{B} > 0. \tag{3.29}$$

The imaginary part of τ at which the lines meet is given by

$$\tau_2^* = \frac{B'D - BD'}{B'C - BC'} = \frac{\mathrm{Im}(Q_2' \cdot wP \cdot \bar{w})}{|P \cdot w|^2} < 0. \tag{3.30}$$

Note that this is the point at which the central charge Z_2' vanishes and it lies in the lower half of the τ plane. Therefore we obtain the condition that the wall of marginal stability is physical if and only if the coefficient of τ_2 in (3.29) is positive

$$\frac{B'C - BC'}{B} = \frac{|Q_1' \cdot w|^2 \mathrm{Im}(P \cdot wQ_2' \cdot \bar{w})}{\mathrm{Im}(Q_1' \cdot wP \cdot \bar{w})} > 0. \tag{3.31}$$

Finally we look at the case $P_1' = 0$ which implies $A = 0$ and $B = \mathrm{Im}(Q_1' \cdot wP \cdot \bar{w}) = 0$. For generic values of w moduli, this can occur only when $Q_1' = \alpha P$ for $\alpha \neq 0$ since for $\alpha = 0$, there is no decay. Then the equation of the circle (3.14) reduces to

$$\alpha |P \cdot w|^2 \tau_2 - \alpha \mathrm{Im}(Q \cdot wP \cdot \bar{w}) = 0. \tag{3.32}$$

Since $\mathrm{Im}(Q \cdot wP \cdot \bar{w}) < 0$, this equation has no solution in the physical upper half τ plane.

We have summarized all the conditions necessary for the occurrence of a physical wall of marginal stability in table 1. For all the cases the condition in (3.17) is neccessary. In what follows we will assume that the moduli w for the general decay always satisfies these conditions. In what follows we will assume that the moduli w for this general decay always satisfies these conditions.

3.5 Intersection of walls of small black hole decays and a generic decay

Since a generic decay is a circle seen in the τ plane, the walls of two decays intersect at the most twice. We have seen that all walls meet at a common point in the lower half plane. Using this fact it is easy to find the other possible point of intersection. In this section we will find this second point of intersection, for a wall corresponding to a small black hole decay and a generic decay. The generic decay can include decays to large black

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holes. Let us denote the second decay by that given in (3.13). To find the intersection point of the circle (3.14) and that corresponding to the small black hole decay in (3.7). It is convenient to parametrize the upper half τ plane using the coordinate τ' given in (3.8). In this coordinate, the equation (3.14) becomes

$$\begin{aligned}&|\tau'|^2\left\{a^2\mathrm{Im}(P_1\cdot wP\cdot\bar{w})-ac\mathrm{Im}(P_1\cdot wQ\cdot\bar{w}+Q_1\cdot wP\cdot\bar{w})+c^2\mathrm{Im}(Q_1\cdot wQ\cdot\bar{w})\right\}\\&\tau_1'\left\{2ab\mathrm{Im}(P_1\cdot wP\cdot\bar{w})-(ad+bc)\mathrm{Im}(P_1\cdot wQ\cdot\bar{w}+Q_1\cdot wP\cdot\bar{w})\right.\\&\left.+2cd\mathrm{Im}(Q_1\cdot wQ\cdot\bar{w})\right\}-\tau_2'\mathrm{Re}(P_1\cdot wQ\cdot\bar{w}-Q_1\cdot wP\cdot\bar{w})\\&+\left\{b^2\mathrm{Im}(P_1\cdot wP\cdot\bar{w})-bd\mathrm{Im}(P_1\cdot wQ\cdot\bar{w}+Q_1\cdot wP\cdot\bar{w})+d^2\mathrm{Im}(Q_1\cdot wQ\cdot\bar{w})\right\}=0.\end{aligned}\tag{3.33}$$

While the small black hole decay reduces to the line given in (3.9). From the discussion in the earlier section 3.1 and from (3.4) we know that, there is always the point

$$\tau^{*\prime}=\frac{Q'\cdot w}{P'\cdot w}=\frac{d\tau^*-b}{-c\tau^*+a},\tag{3.34}$$

at which the line in (3.9) and the circle in (3.33) meet. Here we have just rewritten the point in (3.4) in the τ' co-ordinates. To find the other point we first substitute

$$\tau_1'=\frac{\mathrm{Re}(\tilde{Q}\cdot w\tilde{P}\cdot\bar{w})}{\mathrm{Im}(\tilde{Q}\cdot w\tilde{P}\cdot\bar{w})}\tau_2',\tag{3.35}$$

which arises from the (3.9) into (3.33) to obtain a quadratic equation for the points of intersection. This equation is given by

$$\begin{aligned}&\left(\frac{|\tilde{Q}\cdot w\tilde{P}\cdot\bar{w}|^2}{(\mathrm{Im}(\tilde{Q}\cdot w\tilde{P}\cdot\bar{w}))^2}\right)\left\{a^2\mathrm{Im}(P_1'\cdot wP\cdot\bar{w})-ac\mathrm{Im}(P_1'\cdot wQ\cdot\bar{w}+Q_1'\cdot wP\cdot\bar{w})\right.\\&\qquad\qquad\left.+c^2\mathrm{Im}(Q_1'\cdot wQ\cdot\bar{w})\right\}\tau_2'^2\\&+\left\{\frac{\mathrm{Re}(\tilde{Q}\cdot w\tilde{P}\cdot\bar{w})}{\mathrm{Im}(\tilde{Q}\cdot w\tilde{P}\cdot\bar{w})}\left[2ab\mathrm{Im}(P_1'\cdot wP\cdot\bar{w})-(ad+bc)\mathrm{Im}(P_1'\cdot wQ\cdot\bar{w}+Q_1'\cdot wP\cdot\bar{w})\right.\right.\\&\qquad\left.\left.+2cd\mathrm{Im}(Q_1'\cdot wQ\cdot\bar{w})\right]-\mathrm{Re}(P_1'\cdot wQ\cdot\bar{w}-Q_1'\cdot wP\cdot\bar{w})\right\}\tau_2'\\&+\left\{b^2\mathrm{Im}(P_1'\cdot wP\cdot\bar{w})-bd\mathrm{Im}(P_1'\cdot wQ\cdot\bar{w}+Q_1'\cdot wP\cdot\bar{w})+d^2\mathrm{Im}(Q_1'\cdot wQ\cdot\bar{w})\right\}=0.\end{aligned}\tag{3.36}$$

From the above equation, one can easily read out the product of the two roots, which is given by

$$\begin{aligned}&\frac{b^2\mathrm{Im}(P_1'\cdot wP\cdot\bar{w})-bd\mathrm{Im}(P_1'\cdot wQ\cdot\bar{w}+Q_1'\cdot wP\cdot\bar{w})+d^2\mathrm{Im}(Q_1'\cdot wQ\cdot\bar{w})}{a^2\mathrm{Im}(P_1'\cdot wP\cdot\bar{w})-ac\mathrm{Im}(P_1'\cdot wQ\cdot\bar{w}+Q_1'\cdot wP\cdot\bar{w})+c^2\mathrm{Im}(Q_1'\cdot wQ\cdot\bar{w})}\\&\qquad\times\frac{(\mathrm{Im}(\tilde{Q}\cdot w\tilde{P}\cdot\bar{w}))^2}{|\tilde{Q}\cdot w\tilde{P}\cdot\bar{w}|^2}.\end{aligned}\tag{3.37}$$

Therefore the second root of the quadratic equation is given by

$$\begin{aligned}\tilde{\tau}_2'=&\frac{b^2\mathrm{Im}(P_1'\cdot wP\cdot\bar{w})-bd\mathrm{Im}(P_1'\cdot wQ\cdot\bar{w}+Q_1'\cdot wP\cdot\bar{w})+d^2\mathrm{Im}(Q_1'\cdot wQ\cdot\bar{w})}{a^2\mathrm{Im}(P_1'\cdot wP\cdot\bar{w})-ac\mathrm{Im}(P_1'\cdot wQ\cdot\bar{w}+Q_1'\cdot wP\cdot\bar{w})+c^2\mathrm{Im}(Q_1'\cdot wQ\cdot\bar{w})}\\&\times\frac{\mathrm{Im}(\tilde{Q}\cdot w\tilde{P}\cdot\bar{w})}{|\tilde{Q}\cdot w|^2}.\end{aligned}\tag{3.38}$$

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To obtain the second root we have divided the product of the roots by the imaginary part of $\tau_2^{*\prime}$. Then using the equation of the line (3.35) we can find the real part of the the second point of intersection. This is given by

$$\tilde{\tau}_1' = \frac{b^2\mathrm{Im}(P_1'\cdot wP\cdot\bar{w}) - bd\mathrm{Im}(P_1'\cdot wQ\cdot\bar{w} + Q_1'\cdot wP\cdot\bar{w}) + d^2\mathrm{Im}(Q_1'\cdot wQ\cdot\bar{w})}{a^2\mathrm{Im}(P_1'\cdot wP\cdot\bar{w}) - ac\mathrm{Im}(P_1'\cdot wQ\cdot\bar{w} + Q_1'\cdot wP\cdot\bar{w}) + c^2\mathrm{Im}(Q_1'\cdot wQ\cdot\bar{w})} \times\frac{\mathrm{Re}(\tilde{Q}\cdot w\tilde{P}\cdot\bar{w})}{|\tilde{Q}\cdot w|^2}. \tag{3.39}$$

In general the second point of intersection between a small black hole decay and a generic decay can be in the interior of the upper half plane if $\tilde{\tau}_2' > 0$.

3.6 Walls which never intersect in the interior of the moduli space

In the following we will find the necessary and sufficient conditions such that this second point $\tilde{\tau}'$ never lies in the interior of the upper half plane given any small black hole decay specified by $a, b, c, d, \in \mathbb{Z}$ with $ad-bc = 1$. From (3.38) and the fact that $\mathrm{Im}(Q'\cdot wP'\cdot\bar{w}) < 0$ for all w, we see that the only way $\tilde{\tau}_2' \leq 0$, is to demand

$$R = \frac{\left(b^2\mathrm{Im}(P_1'\cdot wP\cdot\bar{w}) - bd\mathrm{Im}(P_1'\cdot wQ\cdot\bar{w} + Q_1'\cdot wP\cdot\bar{w}) + d^2\mathrm{Im}(Q_1'\cdot wQ\cdot\bar{w})\right)}{\left(a^2\mathrm{Im}(P_1'\cdot wP\cdot\bar{w}) - ac\mathrm{Im}(P_1'\cdot wQ\cdot\bar{w} + Q_1'\cdot wP\cdot\bar{w}) + c^2\mathrm{Im}(Q_1'\cdot wQ\cdot\bar{w})\right)}, \geq 0, \tag{3.40}$$

for all a, b, c, d which satisfies $ad - bc = 1$. We will now show that there are two possible ways to ensure this.

Observe that numerator and the denominator that occurs in the ratio R can be written as

$$R = \frac{v_2^T N v_2}{v_1^T N v_1}, \tag{3.41}$$

where

$$v_2 = \begin{pmatrix} b \\ d \end{pmatrix}, \qquad v_1 = \begin{pmatrix} a \\ c \end{pmatrix} \qquad N = \begin{pmatrix} A & -\frac{B}{2} \\ -\frac{B}{2} & D \end{pmatrix}. \tag{3.42}$$

Now it is clear that if the product of eigen values of the matrix N turns of to be positive then the ratio $R > 0$ for all a, b, c, d. For this to be true we must have the $\mathrm{Det}(N) > 0$. This results in

$$\left[\mathrm{Im}(P_1' wP\cdot\bar{w})\mathrm{Im}(Q_1'\cdot wQ\cdot\bar{w})\right]^2 - \frac{1}{4}[\mathrm{Im}(P_1'\cdot wQ\cdot\bar{w} + Q_1'\cdot wP\cdot\bar{w})]^2 > 0. \tag{3.43}$$

From our discussion in the section 3.3 and from (3.17) we see that the above condition ensures that the wall of marginal stability for the generic decay in (3.14) never intersects the real axis. Further more since it always passes through the point (3.4) in the lower half τ-plane, it lies entirely in the lower half plane. We exclude this situation as the equation (3.14) does not represent a physical wall of marginal stability.

Now if $\mathrm{Det}(N) < 0$, one can factorize the quadratic form

$$Ax^2 - Bxy + Dy^2, \tag{3.44}$$

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where A, B, D are defined in (3.16). We first examine the situation when $A = \mathrm{Im}(P_1' \cdot wP \cdot \bar{w}) \neq 0$, we then can write the above quadratic form as

$$A(x - r_+ y)(x - r_- y), \quad \text{where} \quad r_\pm = \frac{B \pm \sqrt{B^2 - 4AD}}{A}. \tag{3.45}$$

Note that these roots are real as we have $\mathrm{Det}(N) < 0$ and these are the points the the circle (3.14) intersects the real axis. The ratio R for this situation can be written as

$$R = \frac{(b - r_+ d)(b - r_- d)}{(a - r_+ c)(a - r_- c)}, \tag{3.46}$$

We need this ratio $R \geq 0$ for all a, b, c, d with $ad - bc = 1$. We first show that this can be maintained for all a, b, c, d such that $ad - bc = 1$ if r_+ and r_- are rational with the condition

$$r_+ = \frac{p}{q}, \quad r_- = \frac{p'}{q'}, \text{ and } \quad pq' - p'q = 1. \tag{3.47}$$

Here we have chosen $r_+ > r_-$ for definiteness. One can interchange the assignment if $r_- > r_+$. Consider the product of the matrices

$$\begin{pmatrix} q' & -p' \\ -q & p \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{bmatrix} (aq' - p'c) & (bq' - p'd) \\ (-aq + pc) & (-bq + pd) \end{bmatrix}. \tag{3.48}$$

Note that since $pq' - p'q = 1$, the first matrix in the above equation is a $SL(2, \mathbb{Z})$ matrix. Since the product of two $SL(2, \mathbb{Z})$ matrices is also a $SL(2, \mathbb{Z})$ matrix we see that

$$(aq' - p'c)(-bq + pd) - (bq' - p'd)(-aq + pc) = 1. \tag{3.49}$$

Each of the terms in the above equation is an integer, we have the sign of $(aq'-p'c)(-bq+pd)$ and $(bq' - p'd)(-aq + pc)$ is the same or either one of the terms is zero. This implies that the ratio

$$\begin{aligned} R &= \frac{(-bq + pd)(bq' - p'd)}{(-aq + pc)(aq' - p'c)}, \\ &= \frac{(b - \frac{p}{q}d)(b - \frac{p'}{q'}d)}{(a - \frac{p}{q}c)(a - \frac{p'}{q'}c)}, \\ &= \frac{(b - r_+ d)(b - r_- d)}{(a - r_+ c)(a - r_- c)}, \\ &\geq 0, \end{aligned} \tag{3.50}$$

for all $\{a, b, c, d\} \in \mathbb{Z}$ with $ad - bc \in \mathbb{Z}$. From the previous section 3.4, we see that if these decays have to be physical then the coefficients of (3.14) and (3.15) also must satisfy Case (i) of table 1. Note that this basically imposes and inequality on the coefficients $A'B', D$. Since these are conditions on independent coefficients A', B', D', we can always work in the domain of w such that the inequality in (3.22) is satisfied. Note that r_+ and r_- are the points at which the circle (3.14) intersects the real axis in the original τ plane and they agree with the points that the wall corresponding to some small black hole decay intersects

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the real axis given in (3.12). Therefore according to our discussion in section 3.3 the decays which satisfy (3.47) must coincide with the walls corresponding to some small black hole decay. It is important to mention that though the walls of such decays are the same as that of the small black hole decay, the actual decay can be different, since the values r_+ and r_- do not characterize the decay uniquely but only the corresponding wall.

The case when $\mathrm{Det}(N) = 0$ is not of sufficient interest because, in such a situation $r_+ = r_-$ and the wall of marginal stability for the generic decay (3.14) intersects the real line only once from below. It does not emerge in the upper half τ plane and therefore not a physical wall. The only remnant of this wall in the physical τ plane is a point on the real axis.

We finally examine the situation when $A = \mathrm{Im}(P_1' \cdot wP \cdot \bar{w}) = 0$. From table 1 we see that this occurs for $P_1' = \alpha P$ with $0 < \alpha < 1$ or for $P_1' = 0$. This situation is best studied in the original coordinate τ. In the τ plane. The equation corresponding to the wall of marginal stability (3.14) reduces to a line. It intersects the wall corresponding to the small black hole decays at (3.4) in the lower half τ plane. It also intersects the real line at

$$\tau_1 = \frac{D}{B} = \frac{\mathrm{Im}(Q_1' \cdot wQ \cdot \bar{w})}{\mathrm{Im}(P_1' \cdot wQ \cdot \bar{w} + Q_1' \cdot wP \cdot \bar{w})}. \tag{3.51}$$

From the structure of the small black hole decays given in figure 1. it is clear that unless this point in the above equation is an integer, it will intersect one the circles corresponding to the small black hole decay, in particular the small black hole decay corresponding to the bounding circles which join points $(n, 0)$ and $(n + 1, 0)$ on the real line. Therefore we need $\frac{D}{B} = n$ to be an integer to ensure it does not intersect the bounding small black hole decays. When this occurs, a little thought shows that this decay is identical to a small black hole decay characterized by the matrix

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}. \tag{3.52}$$

This is because the line corresponding to the above small black hole decay and the decay with $\frac{D}{B} = n$ intersect at the common point (3.4) and on the real axis at n. Therefore, both of them must be coincident. To summarize we have shown that for $A \neq 0$, a generic decay does not intersect the wall corresponding to the small black hole decay if (3.47) is satisfied. For $A = 0$ one the line corresponding to the generic decay must pass through an integer on the real axis.

We now show that the class of decays which satisfy (3.47) are the only physical decays for which the second point of intersection with the small black hole decays is such that $\tilde{\tau}_2 \leq 0$. That is we prove that the condition in (3.47) is not only necessary but also sufficient. Note that the decays we discussed when $A = 0$ whose walls of marginal stability reduce to straight lines also satisfy the condition (3.47) as one can see from (3.52) Our strategy to prove this will be by elimination. We will show that all the remaining cases are such that one can choose a, b, c, d such that $R < 0$ or equivalently find a small black hole decay which intersects with the generic decay in the interior of the upper half plane. Without loss of generality we will assume that $\frac{p}{q} > \frac{p'}{q'}$ and $q, q' > 0$ with p, q relatively prime, and p', q' relatively prime.

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Case(i) $r_+ = \frac{p}{q}, r_- = \frac{p'}{q'}$ with $pq' - p'q = n$. This case also breaks up into two. Let us first consider the case of $n > qq'$ then $r_+ - r_- > 1$ and there always exists an integer in between r_+ and r_-. From the structure of domains formed by small black hole decays, it can be seen that there is a small black hole decay whose wall is a straight line passing through every integer on the real axis. This line certainly will intersect the circle joining r_+ and r_-. Now consider the case for which $n < qq'$, then by the main theorem of the linear Diophantine equation discussed in the appendix (A.15), we can find $a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$ and $d, c > 0$ such that

$$\begin{aligned} &\frac{b}{d} < \frac{p}{q} < \frac{a}{c}, \qquad \text{with} \qquad d < q, c < q, \\ &pd - bq = 1, \qquad \text{and} \qquad aq - pc = 1, \qquad ad - bc = 1. \end{aligned} \tag{3.53}$$

Now consider the points

$$t_- = \frac{b + mp}{d + mq} < \frac{p}{q} < t_+ = \frac{a - mp}{c - mq}, \tag{3.54}$$

where m is a positive integer. Note that there is a small black hole decay passing through the points t_- and t_+ since $(a - mp)(d + mq) - (b + mp)(c - mq) = 1$. The distance between t_- and r_- is given by

$$t_- - r_- = \frac{n(d + mq) - q'}{qq'(d + mq)}. \tag{3.55}$$

To obtain this we have used the equations in (3.53). By suitably choosing m large enough, it is clear that we can ensure

$$r_- < t_- < r_+ < r_+. \tag{3.56}$$

Now that there is a circle corresponding to small black hole decay joining t_- and t_+, it is clear from the discussion in (3.19) that the circle joining r_- and r_+ intersects the former in the interior of the τ plane. We have therefore seen that if r_+, r_- are rational such that $pq' - p'q = n$ with $n > 1$, there is always a small black hole decay intersecting the wall passing through r_+ and r_- in the interior of the moduli space.

Case(ii) r_+ is irrational. Let us suppose r_+ is irrational. Then by the corollary to the Dirichlet's approximation theorem (A.2) we can find infinite rationals $\frac{p}{q}$ with p and q relatively prime and $q > 0$ such that

$$\left| r_+ - \frac{p}{q} \right| < \frac{1}{q^2}. \tag{3.57}$$

Now using the main theorem on the linear Diophantine equation we know that there exists integers a, b, c, d with $ad - bc = 1$ satisfying (3.53). There is a wall corresponding to a small black hole decay passing through $s_- = \frac{b}{d}$ and $s_+ = \frac{a}{c}$. Let us consider first the case that $\frac{p}{q} > r_+$. Then

$$\begin{aligned} r_+ - \frac{b}{d} &= -\left(\frac{p}{q} - r_+\right) + \left(\frac{p}{q} - \frac{b}{d}\right), \\ &> -\frac{1}{q^2} + \frac{1}{qd} = \frac{q - d}{qd}. \end{aligned} \tag{3.58}$$

Where we have used the inequality (3.57). Now since we have $q > d$, we can conclude that

$$r_- < s_- < r_+ < \frac{p}{q} < s_+. \tag{3.59}$$

From (3.19) we have, the circle corresponding to the small black hole decay joining s_- and s_+ intersects the circle joining r_- and r_+. If $\frac{p}{q} < r_+$, then

$$\begin{aligned} \frac{a}{c} - r_+ &= \left(\frac{a}{c} - \frac{p}{q}\right) + \left(\frac{p}{q} - r_+\right), \\ &> \frac{q-c}{cq}. \end{aligned} \tag{3.60}$$

where we have used the inequality (3.57). Since $q > c$ we conclude that

$$r_- < s_- < \frac{p}{q} < r_+ < \frac{a}{c}. \tag{3.61}$$

Again we have the situation that the circle corresponding to the small black hole decay joining s_- and s_+ intersects the circle joining r_- and r_+. Thus if r_+ is irrational then the wall of marginal stability which passes through r_- and r_+ intersects some wall corresponding to a small black hole decay in the interior of the moduli space.

Case(iii)r_- is irrational. For this case, the argument to show that there exists a wall corrresponding to a small black hole decay which intersects the wall of marginal stability joining r_- and r_+ is same as for the Case(ii) discussed earlier.

Case(iv) r_+ and r_- are irrational. It is clear one can again use the argument for either Case(ii) or Case(iii) to show that there exists a wall corrresponding to a small black hole decay which intersects the wall of marginal stability joining r_- and r_+.

We now have exhausted all the possibilities for the values of r_+ and r_- and have shown that except for the situation when r_- and r_+ are rational and $pq' - p'q = 1$ there is always a small black hole decay which intersects the circle joining r_- and r_+ in the interior of the upper half plane. Thus the necessary and sufficient condition that the walls of marginal stability never intersect in the interior of the upper half τ plane is when the walls satisfy (3.47)

For completeness let us now find the point $\tilde{\tau}_2$ for the class of decays which satisfy (3.47). We have shown that the second point of intersection of this class of black holes and the class of small black holes $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is determined by (3.38). Since $R \geq 0$, the only physically relevant point is when R vanishes or R is ∞. From the expression of R given in (3.40) we see that R can vanish at

$$\frac{p}{q} = \frac{b}{d} \quad \text{or} \quad \frac{p'}{q'} = \frac{b}{d} \tag{3.62}$$

In, this case, $\tilde{\tau}_2' = 0, \tilde{\tau}_1' = 0$ and the intersection point in the original variables is at $\tilde{\tau} = \frac{b}{d}$. R can also be infinity when

$$\frac{p}{q} = \frac{a}{c} \quad \text{or} \quad \frac{p'}{q'} = \frac{b}{d}. \tag{3.63}$$

In this case $\tilde{\tau}'$ is at $i\infty$, while in the original τ variable, the intersection point is at $\tau = \frac{a}{c}$. Thus we can conclude the wall of marginal stability of a small black hole decay intersects with the those which satisfy the condition (3.47) only if the following sets have an overlap

$$\left\{\frac{a}{c}, \frac{b}{d}\right\}, \quad \left\{\frac{p}{q}, \frac{p'}{q'}\right\}. \tag{3.64}$$

Note that this is the same conditions obtained by [9] for the case of small black hole decays in $\mathcal{N} = 4$ theories. This is to be expected since we have seen that the walls which satisfy (3.47) are coincident with small black hole decays.

3.7 Walls bounded by walls of small black hole decays

Using our earlier results, it is now easy to find the conditions on the walls so that they are all confined in domain II for figure 1. That is the walls are such that they are bounded by the bounding walls corresponding to small black hole decays. Consider the class of decays with points of intersection on the real axis r_+ and r_- such that they are in the interval $[n, n+1]$ where $n \in \mathbb{Z}$. That is r_+ and r_- are such that

$$n \leq r_+ \leq n+1, \qquad \text{and} \qquad n \leq r_- \leq n+1 \tag{3.65}$$

Then from the structure of the domains of the small black hole decay shown in figure 1. , we see that such decays never intersect the small black hole decay which passes through the points n and $n+1$ in the interior of the upper half plane. This is because they don't satisfy the condition (3.19) required for intersection in the interior of moduli space. Thus they are lie below this small black hole decay and are therefore they are confined to domain II. Note that if $r_+ = n+1, r_- = n$, then the decay we are considering coincides with the small black hole decay. Thus all decays which satisfy (3.65) are bounded by small black hole decays joining the points $(n, 0)$ and $(n+1, 0)$ Therefore if one restricts the moduli and the charges of decay so that they satisfy (3.65) then the region II in figure 1. is entirely free from any decay in this class.

4 Entropy enigma decays

We have seen that in these class of $\mathcal{N} = 2$ models the moduli can easily be parametrized in terms of the complex coordinates τ and w. The the BPS mass formula is simple expression in terms of charges and these moduli. Furthermore all walls of marginal stability are circles or lines in the τ plane. Therefore it is interesting to re-examine the phenomenon of 'Entropy Enigma' found in [4, 25, 26] and see how they occur in the τ plane. Briefly the Entropy enigma decays are BPS decays which occur when the entropy of the products is parametrically larger than the entropy of the parent. We will now enumerate all the possible charge configurations of the decay products which can lead to this situation for the class of $\mathcal{N} = 2$ models discussed in this paper. Consider the decay

$$\begin{aligned}\begin{pmatrix}\Lambda Q\\ \Lambda P\end{pmatrix} &= \begin{pmatrix}Q_1\\ P_1\end{pmatrix} + \begin{pmatrix}Q_1\\ P_2\end{pmatrix}, \\ &= \begin{pmatrix}\frac{\Lambda}{2}Q + \Lambda^2 q\\ \frac{\Lambda}{2}P + \Lambda^2 p\end{pmatrix} + \begin{pmatrix}\frac{\Lambda}{2}Q - \Lambda^2 q\\ \frac{\Lambda}{2}P - \Lambda^2 p\end{pmatrix}.\end{aligned} \tag{4.1}$$

Since the initial dyon is supersymmetric and a large black hole we have the following

$$Q^2 > 0, \quad P^2 > 0, \quad Q^2P^2 > (Q\cdot P)^2, \quad S = \Lambda^2\pi\sqrt{Q^2P^2 - (Q\cdot P)^2}, \tag{4.2}$$

where S is the Hawking-Bekenstein entropy of the black hole. To ensure that the decay products are supersymmetric and are large black holes in the limit $\Lambda \to \infty$ we have to impose the following conditions

$$Q_i^2 > 0, \qquad P_i^2 > 0, \qquad Q_i^2P_i^2 - (Q_i\cdot P_i)^2 > 0$$

with $i = 1, 2$, in the large Λ limit. This leads to the following 3 cases which exhibit the entropy enigma.

1.

$$\begin{aligned}&q^2 > 0, \quad p^2 > 0, \quad q^2p^2 - (q\cdot p)^2 \\ &S_2 = 2\pi\Lambda^4\sqrt{q^2p^2 - (q\cdot p)^2}.\end{aligned} \tag{4.3}$$

Here S_2 refers to the sum of the leading entropy of the products in the $\Lambda \to \infty$ limit. Note that $q^2 > 0, p^2 > 0$ is obtained if one demands $Q_i^2 > 0, P_i^2 > 0$ in the $\Lambda \to \infty$ limit.

2.

$$\begin{aligned}&q^2 > 0,\ p^2 = 0,\ p\cdot P = 0\ q\cdot p = 0,\ q^2P^2 - (Q\cdot p + q\cdot P)^2 > 0 \\ &S_2 = \Lambda^2\pi\sqrt{q^2P^2 - (Q\cdot p + q\cdot P)^2}\end{aligned} \tag{4.4}$$

Note that for this case, $p^2 = 0$ implies that $p\cdot P = 0$ on demanding $P_1^2 > 0$ and $P_2^2 > 0$ in the $\Lambda \to \infty$ limit. Furthermore, demanding that that the decay products are large black holes (i.e $Q_i^2P_i^2 - (Q_i\cdot P_i)^2 > 0$) in the $\Lambda \to \infty$ limit forces $q\cdot p = 0$ and $q^2P^2 - (Q\cdot p + q\cdot P)^2 > 0$.

3.

$$\begin{aligned}&p^2 > 0,\ q^2 = 0,\ q\cdot Q = 0,\ p\cdot q = 0,\ p^2Q^2 - (Q\cdot p + q\cdot P)^2 \\ &S_2 = \pi\Lambda^3\sqrt{p^2Q^2 - (Q\cdot p + q\cdot P)^2}\end{aligned} \tag{4.5}$$

This is a similar situation to that of case 2. Here $q^2 = 0$ implies that $q\cdot Q = 0$ on demanding $Q_1^2 > 0$ and $P_2^2 > 0$ in the $\Lambda \to \infty$ limit. Demanding that the decay products are large black holes in the $\Lambda \to \infty$ limit gives rise to the condition $q\cdot p = 0$ and $p^2Q^2 - (Q\cdot p + q\cdot P)^2 > 0$.

Finally when one imposes the last possible condition $q^2 = 0, p^2 = 0$, then one is forced to set $q \cdot Q = 0, q \cdot P = 0, q \cdot p = 0, \cdot P$ to ensure that all decay products are supersymmetric and are large black holes. Then the leading entropy of the decay products is given by $\pi\lambda^2/2\sqrt{Q^2P^2 - (Q\cdot P)^2}$ which is not parametrically larger than the parent black hole.

We now examine the wall of marginal stability for these decays and show that if the moduli w is generic, that is all the following moduli dependent quantities do not scale with Λ

$$Q\cdot w, P\cdot w, q\cdot w, p\cdot w \sim O(\Lambda^0), \tag{4.6}$$

then entropy enigma decays are not possible. This phenomenon was observed in the specific examples studied in [4, 26] but was not shown in general.[12] The wall of marginal stability for the decay is determined by the equation (3.14) and inequality (3.15). On substitution of the charges of the decay products in (4.1) in these equations we obtain the following

$$\begin{aligned} \tau\bar\tau \mathrm{Im}[p.wP\cdot\bar w] - \tau_1 \mathrm{Im}[(p\cdot w)(Q\cdot\bar w) + (q\cdot wP\cdot\bar w)] \\ -\tau_2 \mathrm{Re}[(p\cdot w)(Q\cdot\bar w) - (q\cdot wP)(\cdot\bar w)] + \mathrm{Im}[(q\cdot w)(Q\cdot\bar w)] = 0, \end{aligned} \tag{4.7}$$

and

$$\frac{1}{4}|Q\cdot w - \tau P\cdot w|^2 - \Lambda^2 |q\cdot w - \tau p\cdot w|^2 > 0. \tag{4.8}$$

As we have seen earlier, the first equation and the equation equation determining the inequality are circles. From (2.30) we see that the two circles intersect at points where the central charges of the decay products vanish. These points are

$$\tau_+ = \frac{(Q+2\Lambda q)\cdot w}{(P+2\Lambda p)\cdot w}, \qquad \tau_- = \frac{(Q-2\Lambda q)\cdot w}{(P-2\Lambda p)\cdot w}. \tag{4.9}$$

Note that since both these points correspond to points at which the central charges of either of the decay products vanish, these points must lie in the lower half τ plane.

We now analyze the various cases and show that entropy enigma decays at generic values of the w moduli there are no lines or marginal stability corresponding to the entropy enigma in the limit $\Lambda \to \infty$.

Generic values of w-moduli: $\mathrm{Im}(p\cdot wP\cdot\bar w) \neq 0$. In the $\Lambda \to \infty$ limit, and at generic values of the w moduli, more specifically when the moduli is such that (4.6) is satisfied, we see that the points of intersection of the circles coincide (4.9) coincide to $O(\Lambda^{-1})$ terms.

$$\tau_\pm|_{\Lambda\to\infty} = \frac{q\cdot w}{p\cdot w}\left[1 \pm \frac{1}{2\Lambda}\left(\frac{Q\cdot w}{q\cdot w} \mp \frac{P\cdot w}{p\cdot w}\right) + O\left(\frac{1}{\Lambda^2}\right)\right]. \tag{4.10}$$

Therefore to the leading approximation in Λ, the two circles (4.7) and (4.8) are tangents to each other. Now we need to see if the points on the circle (4.7) satisfy the inequality (4.8) in the large Λ limit. If at all this wall is physical, it must emerge in the upper half plane and therefore must satisfy the condition in (3.17). Let us suppose it does, then at these points $|q\cdot w - \tau\cdot w|$ is of $O(\Lambda^0)$ and non-zero. The only point it vanishes is at (4.10)

[12] See below equation (8) of [26]

which is in the lower half τplane. If $|q \cdot w - \tau \cdot w|$ is of $O(\Lambda^0)$ then the second term in (4.8) is the dominant term in the $\Lambda \to \infty$ limit and it is clear that it never satisfies the inequality (4.8). Therefore (4.7) can never be a physical wall of marginal stability for generic values of moduli which satisfy (4.6) in the $\Lambda \to \infty$ limit.

Generic values of w-moduli: $\mathrm{Im}(p{\cdot}wP{\cdot}\bar{w}) = 0$. In general the condition $\mathrm{Im}(p{\cdot}wP{\cdot}\bar{w}) = 0$ imposes an additional condition on the moduli space, and therefore the corresponding wall of marginal stability will not be a co-dimension one surface in the moduli space. Therefore the only way $\mathrm{Im}(p \cdot wP \cdot \bar{w})$ can vanish is $p = \alpha P, \alpha \neq 0$. Then the circle (4.7) reduces to the straight line given by

$$-\tau_1 \mathrm{Im}[n(P \cdot wQ \cdot \bar{w}) + (q \cdot wP \cdot \bar{w})] - \tau_2 \mathrm{Re}[n(P \cdot wQ \cdot \bar{w}) - (q \cdot wP)(\cdot \bar{w})] \\ + \mathrm{Im}[q \cdot wQ \cdot \bar{w}] = 0, \tag{4.11}$$

while the inequality in (4.8) reduces to

$$\frac{1}{4}|Q \cdot w - \tau P \cdot w|^2 - \Lambda^2 |q \cdot w - n\tau P \cdot w|^2 > 0. \tag{4.12}$$

Again the line in (4.11) and the circle determining the inequality in (4.12) intersect each other at the point where the central charges vanish. This is given by

$$\tau_{\pm} = \frac{(Q \pm 2\Lambda q) \cdot w}{P \cdot w(1 \pm 2\alpha\Lambda)}, \\ = \frac{q \cdot w}{\alpha P \cdot w} + O\left(\frac{1}{\Lambda}\right). \tag{4.13}$$

These points must lie in the lower half τ plane even in the $\Lambda \to \infty$ limit.[13] For generic values of the moduli w, that is when (4.6) is satisfied, the part of the line (4.11) which emerges in the upper half plane is such that $|q \cdot w - n\tau P \cdot w| \sim O(\Lambda^0)$. This is because this terms vanishes only at $\tau_\pm$ in the $\Lambda \to \infty$ limit and these point lie in the lower half τ plane. This implies the second term in (4.12) is always dominant in the putative wall of marginal stability (4.11) which emerges in the upper half plane. Thus the inequality (4.12) can never be satisfied and (4.11) does not represent a physical wall of marginal stability at generic values of moduli w.

5 Conclusions

We have studied various properties of walls of marginal stability in $\mathcal{N} = 2$ models with the moduli space given by (1.1). To study these properties we have used the mass formula for BPS states obtained from the classical moduli space of these theories. The list of the properties we have found are listed in the introduction. Using these properties we have isolated a class of decays with walls which always lie in the region bounded by small black

[13]Note that $\tau_\pm = (Q \pm 2\Lambda q) \cdot w / P \cdot w$ for $\alpha = 0$. Thus in $\Lambda \to \infty$ limit it is impossible to ensure that both these points lie in the lower half τ plane for generic values of w. Therefore for generic values of w such decays are not BPS

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hole decays. These walls always lie in region II of figure 1. We hope these properties will be useful in constructing and also testing possible proposals for BPS spectrum in these models.

We showed that these models do not admit entropy enigma decays for generic values of moduli which satisfy (4.6). It will be interesting to find explicit examples of entropy enigma decays by appropriate scaling of the w moduli by the parameter Λ. Such examples can complement the examples found in [4, 26].

Acknowledgments

The author thanks Ashoke Sen for insightful discussions at various stages of this project and also for comments on the manuscript. He thanks Apoorva Patel for a discussion on estimates of real numbers by rationals and providing a useful reference on this topic.

A Some number theory

In this appendix we recall some theorems from number theory which are used in this paper

A.1 Dirichlet's approximation theorem

For each α belonging to the real and N a positive integer, there are $n \in \mathbb{Z}$, $(n \leq N)$ and $p \in \mathbb{Z}$ such that

$$\left|\alpha - \frac{p}{n}\right| < \frac{1}{Nn}, \quad \text{i.e.} \quad |n\alpha - p| < \frac{1}{N} \tag{A.1}$$

Corollary to Dirichlet's approximation theorem. *If α is irrational, then there are infinitely many rationals (with strictly increasing denominators) $\frac{p}{n}$, $n > 0$ with p and n relatively prime such that*

$$\left|\alpha - \frac{p}{n}\right| < \frac{1}{n^2} \tag{A.2}$$

The number $\frac{p}{n}$ is called the D-approximation to α. In fact as an another corollary one can also show that if α is rational, it has only a finite number of D-approximations. For proof of the Dirichlet approximation theorem and its corollaries see [33], see also [34] for a short review.

A.2 Main theorem on the linear Diophantine equation

For each each rational of the form $\frac{p}{q}$ $q > 0$ and p, q relatively prime, there are a, c such that $aq - cp = 1$.

We reproduce the proof of this theorem from [34] with a small modification necessary for our purpose. Let $\alpha = \frac{p}{q}$. If $q = 1$, there $a - cp$ is solved by setting $c = 0$ and $a = 1$. Therefore without loss of generality we may assume that $q \geq 2$. Applying the Dirichlet approximation theorem, with $N = q - 1$, there are $c \in \mathbb{Z}$ and $c > 0, c \leq N = q - 1$ and $a \in \mathbb{Z}$ such that

$$|\alpha c - a| = \left|\frac{p}{q}c - a\right| < \frac{1}{N} = \frac{1}{q-1}, \tag{A.3}$$

multiplying it by q leads to

$$|pc - aq| < \frac{q}{q-1} = 1 + \frac{1}{q-1} \leq 2. \tag{A.4}$$

Since $pc - aq \in \mathbb{Z}$, this implies that $pc - aq| \leq 1$. The case $pc - aq = 0$ is excluded. Because, this implies that

$$\alpha = \frac{p}{q} = \frac{a}{c}. \tag{A.5}$$

and $c \leq N = q - 1 \leq q$. This contradicts the assumption that p and q are relatively prime. Thus the only possibility is $aq - pc = \pm 1$. Let us consider the case

$$aq - pc = 1, \tag{A.6}$$

then we already have the proof of the theorem. In addition to this let us define the integers

$$b = p - a, \qquad d = q - c. \tag{A.7}$$

Note that $q > d > 0$ since $q > c > 0$. Then from (A.6) we see that

$$bq - pd = 1, \qquad ad - bc = 1. \tag{A.8}$$

We therefore have shown the existence of the ratios $\frac{b}{d}$ and $\frac{a}{c}$ in the following order

$$\frac{b}{d} < \frac{p}{q} < \frac{a}{c} \tag{A.9}$$

which satisfies (A.6) and (A.8). Let us now consider the case

$$pc - aq = 1. \tag{A.10}$$

Then we again define the integers

$$b = p - a, \qquad d = q - c, \tag{A.11}$$

again $d > 0$ since $c < q$. From (A.12) we see that

$$bq - pd = 1. \tag{A.12}$$

Thus now we see that the integer b plays the role of a and d plays the role of c required by the theorem. In addition we also have

$$bc - ad = 1. \tag{A.13}$$

which follows from (A.10). Thus in this case we have the ratios in the following increasing order

$$\frac{a}{c} < \frac{p}{q} < \frac{b}{d}, \tag{A.14}$$

satisfying (A.10), (A.12), (A.13)

We have proved the theorem and also shown that given the ratio $\frac{p}{q}$, $q > 0$ and p, q relatively prime, there exists two ratios one lesser and one greater than $\frac{p}{q}$ say $\frac{b}{d}$ and $\frac{a}{c}$ respectively with $q > d > 0, q > c > 0$ such that

$$ad - bc = 1, \qquad pd - bq = 1, \qquad aq - pc = 1. \tag{A.15}$$

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